Math 245B Lecture 3 Notes

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1 Pointwise Convergence, Countability Axioms, Continuity, and Weak Topologies

1.1 The topology of pointwise convergence

Last time we had the following example of a topology.

Example 1.1. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$, for example, $f = C(\mathbb{R})$. For every $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}$, $x_1, \dots, x_m \in \mathbb{R}$, and $\varepsilon > 0$, define $U(t_1, \dots, t_m, x_1, \dots, x_m, \varepsilon) := \{f \in F : |x_i - f(t_i)| < \varepsilon \forall i \leq m\}$. Let $\mathcal{E}$ be the set of all such $U(t_1, \dots, t_m, x_1, \dots, x_m, \varepsilon)$. We claim that if $(f_n)_{n \in \mathbb{N}}$ is a sequence in F, then $f_n \to f$ in $\mathcal{T}(\mathcal{E})$ iff $f_n \to f$ pointwise. Next time, we will show that this topology is not defined by a metric.

Proposition 1.1. *$\mathcal{E}$ is a base for $\mathcal{T}$.*

Proof. We need to check two properties:

1. First, we need $\bigcup \mathcal{E} = F$. Given $f \in F$ and $t_1, \dots, t_m \in \mathbb{R}$, let $x_i = f(t_i)$. Then $f \in U(t_1, \dots, t_m, x_1, \dots, x_m, \varepsilon)$ for all $\varepsilon > 0$.

2. Let $U(t_1, \dots, t_m, x_1, \dots, x_m, \varepsilon), U(s_1, \dots, s_n, y_1, \dots, y_n, \delta) \in \mathcal{E}$. Consider f in their intersection. Choose η so smal that $(f(t_i) - \eta, f(t_i) + \eta) \subseteq (x_i - \varepsilon, x_i + \varepsilon)$ for all i and same for δ. Now $f \in U(t-1, \dots, t_m, s_1, \dots, s_n, f(t_1), \dots, f(t_m), f(s_1), \dots, f(s_n), \eta)$, which is contained in the intersection of the first two sets. $\square$

Proposition 1.2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in F. Then $f_n \to f$ in $\mathcal{T}$ iff $f_n \to f$ pointwise.*

Proof. ($\implies$): Pick $t \in \mathbb{R}$ and $\varepsilon > 0$. Consider $U(t, f(t), \varepsilon)$. There exists n_0 such that $f_n \in U(t, f(t), \varepsilon)$ for all $n \geq n_0$; i.e. $|f(t) - f_n(t)| < \varepsilon$ for all $n \geq n_0$.

($\impliedby$): Let $f \in F$, and let U be a neighborhood of f. Because $\mathcal{E}$ is a base, there exists $U(t_1, \dots, t_m, x_1, \dots, x_m, \varepsilon) \subseteq U$ containing f. By shrinking ε if necessary, we may assume that $x_i = f(t_i)$ for every i. We know that $f_n(t_i) \to f(t_i)$. There exists n_0 such that for all $n \geq n_0$, $|f_n(t_i) - f(t_i)| < \varepsilon$ for all $i \leq m$; i.e. $f_n \in U(t_1, \dots, t_m, x_1, \dots, x_m, \varepsilon)$. $\square$

1.2 Countability axioms and metrizability

Definition 1.1. A topology $\mathcal{T}$ on X is **metrizable** if it is generated by a metric on X.

There are natural and important topologies that are not metrizable. This is why we care about point set topology.

Definition 1.2. A topological space $(X, \mathcal{T})$ is **first countable at** x if it has a countable base at x. The space is **first countable** if it is first countable at every x.

Definition 1.3. A topological space $(X, \mathcal{T})$ is **second countable** if it has a countable base.

Definition 1.4. A topological space $(X, \mathcal{T})$ is **separable** if it has a countable dense subset.

Lemma 1.1. *A metrizable space is first countable.*

Proof. Let ρ generate $\mathcal{T}$. Fix $x \in X$. The collection $\{B(x,r) : r > 0, r \in \mathbb{Q}\}$ is a neighborhood base at x. □

Lemma 1.2. *The topology of pointwise convergence on $\mathbb{R}^{\mathbb{R}}$ is not first countable.*

Proof. Suppose $U_1, U_2, \dots$ contain $f \in \mathbb{R}^{\mathbb{R}}$. We may replace if necessary so that $U_j = U(t_1^{(j)}, \dots, t_m^{(j)}, x_1^{(j)}, \dots, x_m^{(j)}, \varepsilon_j)$. Pick $\varepsilon \neq \infty$, and pick $t \in \mathbb{R} \setminus \{t_i^{(j)} : g \geq 1, i = 1, \dots, m_j\}$. Then $U(t, f(t), \varepsilon)$ is not contained in $U(t_1^{(j)}, \dots, t_m^{(j)}, x_1^{(j)}, \dots, x_m^{(j)}, \varepsilon_j)$ for all j. □

Corollary 1.1. *The topology of pointwise convergence is not metrizable.*

1.3 Continuous functions

Definition 1.5. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$. The function f is **continuous at** x if for every neighborhoddo f V of $f(x)$, there exists a neighborhood U of x such that $f[U] \subseteq V$. f is **continuous** if it is continuous at every point.

Proposition 1.3. *$f : X \to Y$ is continuous if and only if $f^{-1}[U] \in \mathcal{T}_X$ for every $U \in \mathcal{T}_Y$.*

Proof. The same proof from metric spaces works here. □

Proposition 1.4. *If $\mathcal{T}_Y = \mathcal{T}(\mathcal{E})$, then $f : X \to Y$ is continuous if and only if $f^{-1}[U] \in \mathcal{T}_X$ for all $U \in \mathcal{E}$.*

Proof. The proof is the same as for the analogous statement for σ-algebras and measurable functions. □

Definition 1.6. Let $K = \mathbb{R}$ or $\mathbb{C}$, and let $(X, \mathcal{T})$ be a topological space. Then $B(X, K)$ is the set of all **bounded functions** $f : X \to K$. $C(X, K)$ is the set of all **continuous functions** $f : X \to K$. $BC(X, K) = B(X, K) \cap C(X, K)$ is the set of **bounded continuous** functions.

Definition 1.7. On $B(X, K)$ or $BC(X, K)$, the **uniform norm** is $\|f\|_u := \sup_{x \in X} |f(x)|$, and the **uniform metric** is $\rho_u(f, g) := \|f - g\|_u$.

Proposition 1.5. *$BC(X, K)$ is complete with the metric ρ_u.*

1.4 The weak and product topologies

Definition 1.8. Let X be a set, let $((Y_\alpha, \mathcal{T}_\alpha))_{\alpha \in A}$ be topological spaces, and let $f_\alpha : X \to Y_\alpha$ for all $\alpha \in A$. The **weak topology** generated by the f_α is $\mathcal{T}\left(\bigcup_{\alpha \in A}\{f_\alpha^{-1}[U] : U \in \mathcal{T}_\alpha\}\right)$.

Definition 1.9. Let $((Y_\alpha, \mathcal{T}_\alpha))_{\alpha \in A}$ be topological spaces, let $X := \prod_{\alpha \in A} Y_\alpha$, and let $\pi_\alpha : X \to Y_\alpha$ send $(x_\beta)_{\beta \in A} \mapsto x_\alpha$ for all $\alpha \in A$. The **product topology** on X is the weak topology generated by $(\pi_\alpha)_{\alpha \in A}$.

The collection $\{\pi_\alpha^{-1}[U] : \alpha \in A, U \in \mathcal{T}_\alpha\}$ is a subbase for this topology. The collection $\{\bigcap_{j=1}^n \pi_{\alpha_j}^{-1}[U_{\alpha_j}] : \alpha_1, \cdots \alpha_n \in A, U_{\alpha_j} \in \mathcal{T}_{\alpha_j}\}$ is a base for this topology. Our previous topology on $\mathbb{R}^\mathbb{R}$ was actually the product topology.